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LETTER TO THE EDITOR

Nonholonomic mechanics and connections over a bundle map

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Abstract
A general notion of connections over a vector bundle map is considered, and applied to the study of mechanical systems with linear nonholonomic constraints and a Lagrangian of kinetic energy type. In particular, it is shown that the description of the dynamics of such a system in terms of the geodesics of an appropriate connection can be easily recovered within the framework of connections over a vector bundle map. Also the reduction theory of these systems in the presence of symmetry is discussed from this perspective.

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1. Introduction

Inspired by some recent work by Fernandes on connections in Poisson geometry [5] and, more generally, in the context of Lie algebroids [6], we have recently embarked on the study of a general notion of connection, namely connections over a vector bundle map. This new concept covers, besides the standard notions of linear and nonlinear connections, various generalizations such as partial connections and pseudo-connections, as well as the Lie algebroid connections considered by Fernandes. For a detailed treatment we refer to a forthcoming paper, written in collaboration with Cantrijn [2]. After briefly sketching the main idea underlying the notion of connection over a vector bundle map, the purpose of the present letter is to present an application of this theory in the framework of nonholonomic mechanics.

Let M be a real (finite dimensional) C^∞ manifold and $\nu : N \to M$ a vector bundle over M. Assume, in addition, that a linear bundle map $\rho : N \to TM$ is given such that $\tau_M \circ \rho = \nu$, where τ_M denotes the natural tangent bundle projection $TM \to M$. Note that we do not require ρ to be of constant rank. Hence, the image set $\operatorname{Im}\rho$ need not be a vector subbundle of TM but rather determines a generalized distribution as defined by Stefan and Sussmann

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(see, for example, [10, appendix 3]). Denoting the set of (local) sections of an arbitrary bundle E over M by $\Gamma(E)$, it follows that ρ induces a mapping $\Gamma(N) \to \Gamma(TM) = \mathcal{X}(M)$, which we will also denote by ρ. Next, let $\pi : E \to M$ be an arbitrary fibre bundle over M. We may then consider the pull-back bundle $\tilde{\pi}_1 : \pi^*N \to E$ which is a vector bundle. Note that π^*N may be regarded as a fibre bundle over N, with the projection denoted by $\tilde{\pi}_2 : \pi^*N \to N$. A connection on E over ρ or, shortly, *a ρ-connection on E*, is then defined as a linear bundle map $h : \pi^*N \to TE$ from $\tilde{\pi}_1$ to τ_E, over the identity on E, such that, in addition, the following diagram is commutative

$$\begin{array}{ccc} \pi^*N & \xrightarrow{\ h\ } & TE \\ {\scriptstyle \tilde{\pi}_2}\downarrow & & \downarrow{\scriptstyle T\pi} \\ N & \xrightarrow[\ \rho\ ]{} & TM \end{array}$$

(where $T\pi$ denotes the tangent map of π). The image set $\operatorname{Im} h$ determines a generalized distribution on E which projects onto $\operatorname{Im}\rho$. It is important to note that $\operatorname{Im} h$ may have a nonzero intersection with the bundle VE of π-vertical tangent vectors to E. The standard notion of connection is recovered when putting $N = TM$, $\nu = \tau_M$ and ρ the identity map. In case P is a principal G-bundle over M, with right action $R : P \times G \to P$, $(e, g) \mapsto R(e, g) = R_g(e) (= eg)$, a ρ-connection h on P will be called a *principal ρ-connection* if, in addition, it satisfies

$$TR_g(h(e, n)) = h(eg, n)$$

for all $g \in G$ and $(e, n) \in \pi^*N$. Slightly modifying the construction described by Kobayashi and Nomizu [7], given a principal ρ-connection on P, one can construct a ρ-connection on any associated fibre bundle E.

Assume E is a vector bundle and let $\{\phi_t\}$ denote the flow of the canonical dilation vector field on E. A ρ-connection h is then called *a linear ρ-connection* on E if

$$T\phi_t(h(e, n)) = h(\phi_t(e), n)$$

for all $(e, n) \in \pi^*N$. In [2] it is shown that such a linear ρ-connection can be characterized by a mapping $\nabla : \Gamma(N) \times \Gamma(E) \to \Gamma(E)$, $(s, \sigma) \mapsto \nabla_s\sigma$ such that the following properties hold:

(i) ∇ is $\mathbb{R}$-linear in both arguments;
(ii) ∇ is $C^\infty(M)$-linear in s;
(iii) for any $f \in C^\infty(M)$ and for all $s \in \Gamma(N)$ and $\sigma \in \Gamma(E)$ one has $\nabla_s(f\sigma) = f\nabla_s\sigma + (\rho \circ s)(f)\sigma$.

It immediately follows that $\nabla_s\sigma(m)$ only depends on the value of s at m, and therefore we may also write it as $\nabla_{s(m)}\sigma$. Clearly, ∇ plays the role of the covariant derivative operator in the case of an ordinary linear connection. Henceforth, we will also refer to ∇ as a linear ρ-connection. Let k and ℓ denote the fibre dimensions of N and E, respectively, and let $\{s^\alpha : \alpha = 1, \ldots, k\}$, resp. $\{\sigma^A : A = 1, \ldots, \ell\}$, be a local basis of sections of ν, resp. π, defined on a common open neighbourhood $U \subset M$. Then we have $\nabla_{s^\alpha}\sigma^A = \Gamma^{A\alpha}_B\sigma^B$, for some functions $\Gamma^{A\alpha}_B \in C^\infty(U)$, called the connection coefficients of the given ρ-connection.

In order to associate a notion of parallel transport with linear ρ-connections, we first need to introduce a special class of curves in N. A smooth curve $\tilde{c} : I \to N$, defined on a closed interval $I \subset \mathbb{R}$, is called *admissible* if for all $t \in I$, one has $\dot{c}(t) = (\rho \circ \tilde{c})(t)$, where $c = \nu \circ \tilde{c}$

is the projected curve on M. Curves in M that are projections of admissible curves in N are called *base curves*. Note that, in principle, a base curve may reduce to a point.

As in standard connection theory, with any linear ρ-connection ∇ on a vector bundle $\pi : E \to M$, and any admissible curve $\tilde{c} : [a, b] \to N$, one can associate an operator $\nabla_{\tilde{c}}$, acting on sections of π defined along the base curve $c = \nu \circ \tilde{c}$. More precisely, let σ be such a section, i.e. $\sigma : [a, b] \to E$ with $\pi \circ \sigma = c$, then we may put $(\nabla_{\tilde{c}}\sigma)(t) = \nabla_{\tilde{c}(t)}\sigma$ for all $t \in [a, b]$. A section σ, defined along the base curve of an admissible curve $\tilde{c}$, will be called parallel along $\tilde{c}$ if $\nabla_{\tilde{c}(t)}\sigma = 0$ for all t. In coordinates, this yields a system of linear differential equations for the components of σ and, again using standard arguments, one can show that this leads to a notion of parallel transport on E along admissible curves in N (cf [2] for more details).

As an application of the above formalism, we will consider a mechanical system consisting of a free particle subjected to some linear nonholonomic constraints.

2. Nonholonomic mechanics

Let g be a Riemannian metric on an n-dimensional manifold M. Consider a free particle, with configuration space M and Lagrangian $L : TM \to \mathbb{R}, v \mapsto L(v) = 1/2g(v, v)$. It is well known that the equation of motion can be written as the geodesic equation $\nabla^g_{\dot{c}}\dot{c}(t) = 0$, where ∇^g is the Levi-Civita connection corresponding to g. Suppose now that the system is subjected to $n - k$ (independent) linear nonholonomic constraints, defining a regular non-integrable k-dimensional distribution Q on M. We then have a direct sum decomposition $TM = Q \oplus Q^\perp$, where $Q^\perp$ is the orthogonal complement of Q with respect to the given metric g. The projections of TM onto Q and $Q^\perp$ will be denoted by π_Q and $\pi^\perp_Q$, respectively. It is well known that the solution curves of the nonholonomic free particle are curves c in M satisfying the equation $\pi_Q(\nabla^g_{\dot{c}}\dot{c}(t)) = 0$, together with the constraint condition $\dot{c}(t) \in Q$ for all t (see, for instance, [9]). Furthermore, one can define a linear connection $\bar{\nabla}$ on M according to $\bar{\nabla}_X Y = \nabla^g_X Y + (\nabla^g_X \pi_{Q^\perp})(Y)$ for $X, Y \in \mathcal{X}(M)$. This connection restricts to Q and the equation of motion of the nonholonomic free particle can be rewritten as $\bar{\nabla}_{\dot{c}}\dot{c}(t) = 0$, with initial velocity taken in Q (see [1, 9]).

We now reconsider the nonholonomic free particle from the point of view of connections over a vector bundle map. Let $i : Q \to TM$ denote the natural embedding of Q into TM. In the sequel we will identify $X \in \Gamma(Q)$ with $Ti \circ X$, regarded as a vector field on M. In terms of the notations used above, we consider the following situation: $N = E = Q$, $\nu = \pi = (\tau_M)_{|Q}$ and $\rho = i$. We may now define a linear connection $\nabla^{\mathrm{nh}} : \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q)$ over i on the vector bundle $\pi : Q \to M$ by the prescription

$$\nabla^{\mathrm{nh}}_X Y = \pi_Q \nabla^g_X Y$$

where the superscript 'nh' stands for 'nonholonomic'. It is easily seen that this indeed determines a linear i-connection and that, moreover, $\nabla^{\mathrm{nh}}_X Y = \bar{\nabla}_X Y$ for $X, Y \in \Gamma(Q)$. Admissible curves in this setting are curves $\tilde{c}$ in Q that are prolongations of curves in M, i.e. $\tilde{c}(t) = \dot{c}(t)$ for some curve c in M. Note that for any base curve c, $\dot{c}$ may be regarded here both as an admissible curve in Q and as a section of π defined along c. It follows that the equation of motion of the given nonholonomic problem can be written as $\nabla^{\mathrm{nh}}_{\dot{c}}\dot{c}(t) = 0$, where c is a curve in M tangent to Q.

The restriction of the given Riemannian metric g on M to sections of Q defines a bundle metric on Q which we denote by g^{o}. The i-connection ∇^{nh} considered above now admits the following characterization.

Proposition 1. $\nabla^{\rm nh}$ *is uniquely determined by the conditions that it is 'metric', i.e. for all* $X, Y, Z \in \Gamma(Q)$ *one has*

$$X(g^{\rm o}(Y, Z)) = g^{\rm o}(\nabla^{\rm nh}_X Y, Z) + g^{\rm o}(Y, \nabla^{\rm nh}_X Z)$$

and that it satisfies

$$\nabla^{\rm nh}_X Y - \nabla^{\rm nh}_Y X = \pi_Q[X, Y]$$

for all $X, Y \in \Gamma(Q)$.

Proof. First we prove that $\nabla^{\rm nh}$ satisfies both conditions. Using the fact that ∇^g is metric for g, and regarding sections of Q as vector fields on M, we find

$$\begin{aligned} X(g^{\rm o}(Y, Z)) &= X(g(Y, Z)) \\ &= g(\nabla^g_X Y, Z) + g(Y, \nabla^g_X Z) \\ &= g^{\rm o}(\nabla^{\rm nh}_X Y, Z) + g^{\rm o}(Y, \nabla^{\rm nh}_X Z) \end{aligned}$$

where the last equality follows from the fact that $g(X, Y) = 0$ whenever $X \in \Gamma(Q)$ and $Y \in \Gamma(Q^\perp)$. The second condition follows from the symmetry property of ∇^g (i.e. ∇^g has zero torsion).

Conversely, let ∇ be an arbitrary linear i-connection that satisfies both conditions. One then easily derives that for any chosen $X, Y \in \Gamma(Q)$ and all $Z \in \Gamma(Q)$

$$\begin{aligned} 2g^{\rm o}(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) \\ &\quad + g(\pi_Q[X, Y], Z) - g(\pi_Q[X, Z], Y) - g(X, \pi_Q[Y, Z]) \\ &= 2g(\nabla^g_X Y, Z) \end{aligned}$$

from which one readily deduces that $\nabla_X Y = \pi_Q \nabla^g_X Y$, i.e. $\nabla \equiv \nabla^{\rm nh}$. □

It is easily proven that if Q is an integrable distribution defining a foliation of M (i.e. the given constraints are holonomic), then the connection $\nabla^{\rm nh}$ induces the Levi-Civita connection on the leaves of this foliation with respect to the induced metric.

From the fact that the nonholonomic connection $\nabla^{\rm nh}$ is metric it follows that for any $X, Y \in \Gamma(Q)$

$$X(g^{\rm o}(X, Y)) = g^{\rm o}(\nabla^{\rm nh}_X X, Y) + g^{\rm o}(X, \nabla^{\rm nh}_X Y).$$

The second term on the right-hand side can be rewritten as

$$\begin{aligned} g^{\rm o}(X, \nabla^{\rm nh}_X Y) &= g^{\rm o}(X, \nabla^{\rm nh}_Y X) + g(X, [X, Y]) \\ &= \tfrac{1}{2}\mathcal{L}_Y(g(X, X)) + g(X, [X, Y]) \\ &= \tfrac{1}{2}(\mathcal{L}_Y g)(X, X) \end{aligned}$$

where $\mathcal{L}$ denotes the Lie derivative operator. With any given $Y \in \Gamma(Q)$ one can associate a function J_Y on Q, given by $J_Y(X_m) := g^{\rm o}(X_m, Y(m))$, for all $m \in M$ and $X_m \in Q_m$. Using the preceding identities, and considering a base curve c in M which is 'geodesic' with respect to $\nabla^{\rm nh}$ (i.e. a solution of the nonholonomic equations), one easily derives that

$$\frac{\rm d}{{\rm d}t}(J_Y(\dot c))(t) = \frac{1}{2}(\mathcal{L}_Y g)(\dot c(t), \dot c(t)).$$

This equation implies that every section Y of Q which, regarded as a vector field on M, leaves the metric g invariant (i.e. is a Killing vector field) determines a conserved quantity for the given nonholonomic system.

3. Reduction of the nonholonomic free particle with symmetry

Let G be a Lie group defining a free and proper right action on M, denoted by $R_a : M \to M, m \mapsto R_a(m) = ma$, for all $a \in G$, such that we have a principal fibre bundle $M \stackrel{\mu}{\to} \hat{M} := M/G$. Assume this action leaves invariant both the Riemannian metric g and the constraint distribution Q, i.e. $R_a^* g = g$ and $TR_a(Q) \subset Q$ for all $a \in G$. We already know from above that the equations of motion of the nonholonomic free particle are given by the 'geodesic' equations: $\nabla^{\mathrm{nh}}_{\dot{c}} \dot{c}(t) = 0$. Using the symmetry assumption (i.e. the G-invariance of g and Q), it is easily proven that if $c(t)$ is a solution, so is $c(t)a$ for all $a \in G$. Therefore, one obtains equivalence classes of solutions, where two solutions c_1 and c_2 are called equivalent iff $c_1 = c_2 a$ for some $a \in G$. In the reduction procedure described below, it is our intention to construct a reduced connection over a suitable vector bundle map, such that the corresponding 'geodesics' are precisely these equivalence classes.

First of all, we note that the set Q/G, the quotient space of Q under the lifted action of G on Q, admits a vector bundle structure over $\hat{M}$, with projection $\tau : Q/G \to \hat{M}$ defined by $\tau([X_m]) = \mu(m)$. Here, $[X_m]$ represents the G-orbit of $X_m \in Q$ under the lifted right action. Using the fact that this action on Q is fibre linear, and relying on the local triviality of the principal bundle $M \to \hat{M}$, one can verify that τ indeed determines a vector bundle structure (see, for example, [11, p 29]). Next, we define a map $\rho : Q/G \to T\hat{M}$ according to $\rho([X_m]) := T\mu(X_m)$. Once again one can easily see that this map is well defined (i.e. does not depend on the chosen representative X_m of $[X_m]$) and is fibred over the identity on $\hat{M}$. We now first construct a principal ρ-connection on M which, subsequently, will be used to define a linear ρ-connection on Q/G.

Let $h : \mu^*(Q/G) \to TM : (m, [X_m]) \to X_m$, i.e. we take the image $h(m, [X_m])$ to be the unique tangent vector at m belonging to the equivalence class $[X_m]$. Since the action of G is free, it immediately follows that h is well defined and, moreover, $\operatorname{Im} h = Q$. We can also verify that $h(ma, [X_m]) = TR_a(X_m) = TR_a(h(m, [X_m]))$ and $T\mu(h(m, [X_m])) = \rho([X_m])$. Consequently, h determines a principal ρ-connection on M (see the definition above).

Note that sections of the bundle $\tau : Q/G \to \hat{M}$ can be put into one-to-one correspondence with the set of right invariant vector fields on M taking values in Q (i.e. the right equivariant sections of $Q \to M$). Indeed, for $\psi \in \Gamma(Q/G)$ and $m \in M$ such that $\mu(m) \in \operatorname{dom} \psi$, put

$$\psi^h(m) := h(m, \psi(\mu(m))).$$

Then ψ^h is a G-equivariant section of Q. On the other hand, let X be a right invariant vector field on M with values in Q. Then, define an element X_h of $\Gamma(Q/G)$ by

$$X_h(\hat{m}) = [X_m]$$

with $m \in \mu^{-1}(\hat{m})$. Clearly, this does not depend on the choice of m in the fibre over $\hat{m}$. Thus, by means of h we have established a bijective correspondence between $\Gamma(Q/G)$ and the set of G-equivariant sections of $Q \to M$. For the following derivation of a reduced ρ-connection on Q/G, we may refer to Cantrijn *et al* [1] where, at least for the so-called Chaplygin case, a similar construction has been made in terms of 'ordinary' connections and, therefore, we will not enter into details. For completeness, however, we recall the following useful properties. Firstly, from the G-invariance of g one can deduce that the vector field $\nabla^g_X Y$ is right invariant whenever $X, Y \in \mathcal{X}(M)$ are right invariant, and that $\pi_Q : TM \to Q$ commutes with TR_a for any $a \in G$. Secondly, the symmetry assumptions also imply that the induced bundle metric g^{o} on Q is G-invariant and, hence, determines a reduced bundle metric $\hat{g}^{\mathrm{o}}$ on Q/G. Using h we can construct $\hat{g}^{\mathrm{o}}$ as follows: for any $\phi, \psi \in \Gamma(Q/G)$ put

$$\hat{g}^{\mathrm{o}}(\hat{m})(\phi(\hat{m}), \psi(\hat{m})) := g^{\mathrm{o}}(m)(\phi^h(m), \psi^h(m))$$

with $m \in \mu^{-1}(\hat{m})$. Let $a \in G$, then

$$\begin{aligned} g^{\mathrm{o}}(ma)(\phi^h(ma), \psi^h(ma)) &= g(ma)(TR_a\phi^h(m), TR_a\psi^h(m)) \\ &= g^{\mathrm{o}}(m)(\phi^h(m), \psi^h(m)) \end{aligned}$$

where, again, we have relied on the G-invariance of g. From this we may conclude that $\hat{g}^{\mathrm{o}}$ is indeed well defined.

Let ∇^{nh} be the nonholonomic connection over i, introduced in the previous section. We now construct a linear ρ-connection on the bundle Q/G, as follows: for any $\psi, \phi \in \Gamma(Q/G)$ put

$$\hat{\nabla}^{\mathrm{nh}}_{\psi}\phi = (\nabla^{\mathrm{nh}}_{\psi^h}\phi^h)_h.$$

Again, one may check that this is well defined and verifies the conditions of a linear ρ-connection.

Proposition 2. *The linear ρ-connection $\hat{\nabla}^{\mathrm{nh}}$ is metric with respect to the reduced bundle metric $\hat{g}^{\mathrm{o}}$ on Q/G, and satisfies the property*

$$\hat{\nabla}^{\mathrm{nh}}_{\psi}\phi - \hat{\nabla}^{\mathrm{nh}}_{\phi}\psi - [\psi, \phi] = 0$$

where, by definition, $[\psi, \phi] := (\pi_Q[\psi^h, \phi^h])_h$.

Proof. For any $\psi \in \Gamma(Q/G)$, we have that ψ^h is μ-related to $\rho \circ \psi$ as vector fields on M and $\hat{M}$, respectively. Using this, together with the properties of ∇^{nh}, we can prove that $\hat{\nabla}^{\mathrm{nh}}$ is metric with respect to $\hat{g}^{\mathrm{o}}$. Indeed, let $\psi, \phi, \eta \in \Gamma(Q/G)$, then

$$\begin{aligned} (\rho \circ \psi)(\hat{g}^{\mathrm{o}}(\phi, \eta)) \circ \mu &= \psi^h(\hat{g}^{\mathrm{o}}(\phi, \eta) \circ \mu) \\ &= \psi^h(g^{\mathrm{o}}(\phi^h, \eta^h)) \\ &= g^{\mathrm{o}}(\nabla^{\mathrm{nh}}_{\psi^h}\phi^h, \eta^h) + g^{\mathrm{o}}(\phi^h, \nabla^{\mathrm{nh}}_{\psi^h}\eta^h) \\ &= \big(\hat{g}^{\mathrm{o}}(\hat{\nabla}^{\mathrm{nh}}_{\psi}\phi, \eta) + \hat{g}^{\mathrm{o}}(\phi, \hat{\nabla}^{\mathrm{nh}}_{\psi}\eta)\big) \circ \mu \end{aligned}$$

from which the result readily follows.

The second property can also be proven in a straightforward manner. □

It is also not difficult to verify that $\hat{\nabla}^{\mathrm{nh}}$ is uniquely determined by the two properties mentioned in the proposition.

To complete the reduction picture, it can be proven that every solution of the geodesic equation for ∇^{nh} projects onto a solution of the 'geodesic problem' for the reduced nonholonomic connection $\hat{\nabla}^{\mathrm{nh}}$ in the following sense. Assume that c is a solution of the nonholonomic equations, i.e. $\nabla^{\mathrm{nh}}_{\dot{c}}\dot{c}(t) = 0$. Consider the curve $\hat{c} = \mu \circ c$ in $\hat{M}$. Then the section $[\dot{c}](t) = [\dot{c}(t)]$ of Q/G along $\hat{c}$ is autoparallel with respect to the ρ-connection $\hat{\nabla}^{\mathrm{nh}}$, i.e. $\hat{\nabla}^{\mathrm{nh}}_{[\dot{c}]}[\dot{c}](t) = 0$. This follows from the fact that for each $m \in M$, $h(m, .) : \tau^{-1}(\mu(m)) \to T_mM$ is injective and that for any base curve c in M

$$h\big(c(t), \hat{\nabla}^{\mathrm{nh}}_{[\dot{c}]}[\dot{c}](t)\big) = \nabla^{\mathrm{nh}}_{\dot{c}}\dot{c}(t) \quad \forall t.$$

On the other hand, any solution $[\dot{c}]$ of the equation $\hat{\nabla}^{\mathrm{nh}}_{[\dot{c}]}[\dot{c}](t) = 0$ determines an equivalence class of solutions of the initial nonholonomic problem on M. Given any point c_0 in $\mu^{-1}(\tau([\dot{c}](0)))$, a unique curve c in M can be constructed which is horizontal with respect to the principal ρ-connection h on M, i.e. c satisfies for all t: $\dot{c}(t) = h(c(t), [\dot{c}](t))$ with the initial condition $c(0) = c_0$ (note that $[\dot{c}(t)] = [\dot{c}](t)$). It is easily seen that $\mu(c) = \tau([\dot{c}])$ and from this we can deduce $\nabla^{\mathrm{nh}}_{\dot{c}}\dot{c}(t) = 0$.

We conclude that the set of equivalence classes of solutions of the free nonholonomic mechanical problem in M is in a one-to-one correspondence with the set of solutions of autoparallel admissible curves with respect to the reduced nonholonomic connection (i.e. using the principal ρ-connection h).

To close this section, we note that much of the preceding discussion can be easily extended to more general nonholonomic systems with symmetry, admitting forces derivable from a G-invariant potential energy function.

4. Final remarks

Our approach to the reduction problem of a nonholonomic free particle with symmetry, using the generalized notion of connections over a bundle map, differs from other approaches in that we do not have to make any assumption regarding the (constant) rank of the constraint distribution Q. In treatments of the so-called Chaplygin case, for instance, the assumption is that Q is the horizontal distribution of a principal G-connection (see, for example, [1, 4, 8]), i.e. besides being G-invariant Q also satisfies $TM = Q \oplus \ker T\mu$. In the more general case treated, for example, by Cendra *et al* [3], it is assumed that $TM = Q + \ker T\mu$ (but one may have $Q \cap \ker T\mu \neq \{0\}$). In our treatment we only require G-invariance of Q.

Finally, in a forthcoming paper devoted to the use of the concept of a connection over a vector bundle map in sub-Riemannian geometry, it will be demonstrated that the above application to nonholonomic mechanics may also shed some new light on the relationship between the so-called 'vakonomic' and the 'nonholonomic' treatment of systems with constraints.

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